Math 250 7.1 - Day 2 Inverse Functions

objectives:

1) Continue practice of finding the slope of tangent line to the inverse function using

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$$

where (y_0, x_0) on f^{-1}
and (x_0, y_0) on f

2) Theorems about
increasing/decreasing of f^{-1}
continuity of f^{-1}
differentiability of f^{-1}

7.2 - Day 0
Background on Logs + Exponentials

Math 250

If a function f is <u>increasing</u>,
then its inverse f^{-1} is also increasing.

If a function f is <u>decreasing</u>
then its inverse f^{-1} is also decreasing.

If a function f is <u>continuous</u>,
then its inverse f^{-1} is also continuous.

If a function f is <u>differentiable</u>,
then its inverse f^{-1} is also differentiable
<u>except</u> if $f'(\ )=0$
then $(f^{-1})'(\ )$ is undefined
and not differentiable.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$$

(y_0, x_0) on f^{-1}	(x_0, y_0) on f
$(f^{-1})'(y_0)$ undefined	$f'(x_0) = 0$
(f^{-1}) is not differentiable at $x = y_0$	f is differentiable at $x = x_0$.

Math 250 5.3

① Find $(f^{-1})'(7)$ when $f(x) = 5 - 2x^3$

step 1: 7 is an x-coord on f^{-1} meaning $f^{-1}(7) = x_0$ and $f(x_0) = 7$.

$5 - 2x_0^3 = 7$

$-2x_0^3 = 2$

$x_0^3 = -1$

$x_0 = -1$ Can be solved w/o GC!

$(7, -1)$ on f^{-1}

$(-1, 7)$ on f

step 2: $f'(x) = -6x^2$

step 3: Evaluate $f'(x)$ using x-coordinate on f.

$f'(-1) = -6(-1)^2 = -6$.

step 4: $(f^{-1})'(7) = \dfrac{1}{f'(f^{-1}(7))} = \dfrac{1}{f'(-1)} = \dfrac{1}{-6} = \boxed{\dfrac{-1}{6}}$

The slope of the tangent line to $f^{-1}(x)$ at $(7, -1)$ is $-\frac{1}{6}$.

The slope of the tangent line to $f(x)$ at $(-1, 7)$ is -6.

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(2) Find $(f^{-1})'(-9)$ when $f(x) = x^3 + x + 1$

step 1: -9 is an x-coordinate on f^{-1} meaning $f^{-1}(-9) = x_0$

and $f(x_0) = -9$

$x_0^3 + x_0 + 1 = -9$ ← can't solve algebraically w/o rational root theorem and/or GC.

$x_0^3 + x_0 + 10 = 0$

$x_0 = -2$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 1 & 10 \\ & & -2 & 4 & -10 \\ \hline & 1 & -2 & 5 & 0 \end{array} \checkmark$$

$(-9, -2)$ on f^{-1}

$(-2, -9)$ on f

or $f(-2) = (-2)^3 + (-2) + 1 = -9 \checkmark$

step 2: $f'(x) = 3x^2 + 1$

step 3: Evaluate f' at x-coord on graph of f.

$$f'(-2) = 3(-2)^2 + 1 = 3(4) + 1 = 13$$

step 4: $(f^{-1})'(-9) = \dfrac{1}{f'(f^{-1}(-9))} = \dfrac{1}{f'(-2)} = \boxed{\dfrac{1}{13}}$

The slope of the tangent line to f^{-1} at $(-9, -2)$ is $\frac{1}{13}$.

The slope of the tangent line to f at $(-2, -9)$ is 13.

Math 250

limit to a portion that passes HLT!

③ Find $(f^{-1})'(1)$ when $f(x) = \cos(2x)$ $\quad \underbrace{0 \le x \le \frac{\pi}{2}}$

step 1: 1 is an x-coord on f^{-1}, meaning $f^{-1}(1) = x_0$.

and $f(x_0) = 1$

$$f(x_0) = \cos(2x_0) = 1$$

(can be solved w/o GC)

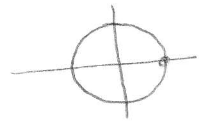
we seek an angle $\theta = 2$ so that $\cos\theta = 1$

$\cos(0) = 1$

$\cos(2\pi) = 1$

$\cos(-2\pi) = 1$

etc.



$\theta = 0$

$2x_0 = 0$

$x_0 = 0.$

$(1, 0)$ on f^{-1}

$(0, 1)$ on f.

chain rule!

step 2: $f'(x) = -\sin(2x) \cdot 2$

$= -2\sin(2x).$

step 3: Evaluate $f'(x)$ using x-coordinate on graph of f.

$f'(0) = -2\sin(2\cdot 0)$

$= -2\sin(0)$

$= -2\cdot 0$

$= 0.$

step 4: $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{0} = \boxed{\text{undefined}}$

The slope of the tangent line to $f^{-1}(x)$ at $x=1$ is undefined.

(vertical tangent line to $f^{-1}(x)$ at $(1, 0)$.)

slope of tangent line to $f(x)$ at $x=0$ is 0.

(horizontal tangent line to $f(x)$ at $(0, 1)$.)

f^{-1} is not differentiable at $x=1$.

f is differentiable at $x=0$. (But $x=0$ is a C.V. for f.)

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(4) Find $(f^{-1})'(-31)$ when $f(x) = -x^3 - x - 1$ $\quad (y_0 = -31)$

Step 1: -31 is an x-coord on graph of f^{-1}, so $f^{-1}(-31) = x_0$

and $f(x_0) = -31$

$$-x_0^3 - x_0 - 1 = -31$$

$$0 = x_0^3 + x_0 - 30$$ ← can't be solved w/o Rational Root Theorem and/or GC.

$x_0 = 3$ from graph

3	1	0	1	-30
		3	9	30
	1	3	10	0 ✓

or $f(3) = -3^3 - 3 - 1 = -31$ ✓

$(3, -31)$ on f

$(-31, 3)$ on f^{-1}

Step 2: $f'(x) = -3x^2 - 1$

Step 3: Evaluate $f'(x)$ using an x-coordinate on graph of f.

$$f'(3) = -3(3)^2 - 1$$
$$= -3(9) - 1$$
$$= -28$$

Step 4: $$(f^{-1})'(-31) = \frac{1}{f'(f^{-1}(-31))} = \frac{1}{f'(3)} = \frac{1}{-28} = \boxed{\frac{-1}{28}}$$

The slope of tangent line to f^{-1} at $(-31, 3)$ is $-\frac{1}{28}$.

The slope of tangent line to f at $(3, -31)$ is -28.

Math 250

(5) Find $\frac{dy}{dx}$ at $(0,2)$ when $x = 2\ln(y^2-3)$.

Method 1: Use implicit differentiation.

$$x = 2\ln(y^2-3)$$

$$1 = 2 \cdot \frac{1}{y^2-3} \cdot 2y\frac{dy}{dx}$$

$$1 = \frac{4y}{y^2-3} \cdot \frac{dy}{dx}$$

$$\frac{y^2-3}{4y} = \frac{dy}{dx}$$

Evaluate when $x=0,\ y=2$ where x & y refer to $f(x)$.

$$\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{2^2-3}{4(2)} = \boxed{\frac{1}{8}}$$

Method 2: Find $\frac{dx}{dy}$ and take reciprocal.

$$x = 2\ln(y^2-3)$$

$$\frac{dx}{dy} = 2 \cdot \frac{1}{y^2-3} \cdot 2y$$

$$\frac{dx}{dy} = \frac{4y}{y^2-3}$$

$$\frac{dy}{dx} = \frac{y^2-3}{4y}$$

Evaluate when $x=0,\ y=2$ where x & y refer to $f^{-1}(x)$

$$\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{2^2-3}{4(2)} = \boxed{\frac{1}{8}}$$

Math 250

⑥: Use the derivative of f to determine if f is invertible:

$$f(x) = \frac{x}{\sqrt{x^2+7}} = x\,(x^2+7)^{-1/2}$$

product & chain rules!

$$f'(x) = x \cdot \left(-\tfrac{1}{2}\right)(x^2+7)^{-3/2} \cdot (2x) + (x^2+7)^{-1/2} \cdot 1$$

$$= -x^2(x^2+7)^{-3/2} + (x^2+7)^{-1/2}$$

$$= (x^2+7)^{-3/2}\left[-x + (x^2+7)\right]$$

$$= (x^2+7)^{-3/2}(x^2-x+7)$$

$$= \frac{x^2-x+7}{(\sqrt{x^2+7})^3}$$

$f'(x) = 0$ when numerator $= 0$

$x^2 - x + 7 = 0$ doesn't factor

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(7)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-27}}{2}$$ imaginary

$f'(x)$ undefined when denominator $= 0$.

$(\sqrt{x^2+7})^3 = 0$

$x^2 + 7 = 0$

$x^2 = -7$ imaginary.

No C.Vs.

$f' \longleftrightarrow$ (+)

f increasing $(-\infty, \infty)$

f monotonic

f invertible.

Math 250

7. Find $(f^{-1})'\left(\frac{\sqrt{2}}{4}\right)$ when $f(x)=\dfrac{x}{\sqrt{x^2+7}}$

step 1: $\frac{\sqrt{2}}{4}$ is an x-coord on graph of f^{-1} : $f^{-1}\left(\frac{\sqrt{2}}{4}\right)=x_0$

so $f(x_0)=\dfrac{\sqrt{2}}{4}$

$$\frac{x_0}{\sqrt{x_0^2+7}}=\frac{\sqrt{2}}{4}$$

$$4x_0=\sqrt{2x_0^2+14}$$

$$16x_0^2=2x_0^2+14 \quad \text{* square both sides}$$

$$14x_0^2-14=0$$

$$14(x_0^2-1)=0$$

$$14(x_0-1)(x_0+1)=0$$

$$x_0=1,\ x_0=-1$$

huh? How can we get 2 solutions? We can't, because f(x) is one-to-one. When we squared both sides, we created an extraneous solution!

check $x_0=1$

$$4x_0=\sqrt{2x_0^2+14}$$

$$4(1)\stackrel{?}{=}\sqrt{2(1)^2+14}$$

$$4=\sqrt{2+14}$$

$$4=\sqrt{16} \quad \checkmark \text{ yes.}$$

check $x_0=-1$

$$4(-1)=\sqrt{2(-1)^2+14}$$

$$-4\neq\sqrt{16} \quad \text{X No}$$

So $\left(1,\frac{\sqrt{2}}{4}\right)$ on f

$\left(\frac{\sqrt{2}}{4},1\right)$ on f^{-1}.

step 2: $f'(x)$. See previous question

$$f'(x)=\frac{x^2-x+7}{\left(\sqrt{x^2+7}\right)^3}$$

step 3: Evaluate $f'(x)$ at x-coord for f.

$$f'(1)=\frac{1^2-1+7}{\left(\sqrt{1^2+8}\right)^3}$$

Math 250

7 continued.

$$= \frac{7}{3^3}$$

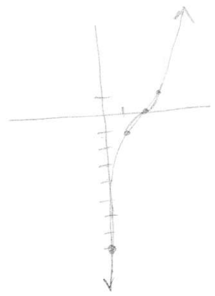
$$= \frac{7}{27}$$

step 4:

$$(f^{-1})'\left(\frac{\sqrt{2}}{4}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{\sqrt{2}}{4}\right)\right)} = \frac{1}{f'(1)} = \frac{1}{(7/27)} = \boxed{\frac{27}{7}}$$

Math 250

(8) Find $(f^{-1})'(x)$ implicitly for $f(x) = (x-2)^3$



one-to-one
has inverse

$$y = (x-2)^3$$

$$\frac{dy}{dx} = 3(x-2)^2$$

$$\frac{dx}{dy} = \frac{1}{3(x-2)^2}$$

(x,y) on $f(x)$
(y,x) on $f^{-1}(x)$

$$\boxed{\frac{dy}{dx} = \frac{1}{3(y-2)^2}}$$

switch (x,y) to (y,x)
(x,y) now on $f^{-1}(x)$

$$\boxed{(f^{-1})'(x) = \frac{1}{3(f^{-1}(x)-2)^2}}$$

(9) Use result to find $(f^{-1})'(2)$.

$$(f^{-1})'(2) = \frac{1}{3(f^{-1}(2)-2)^2}$$

$$f^{-1}(2) = x_0$$

means

$$2 = f(x_0)$$

$$2 = (x_0-2)^3$$

$$\sqrt[3]{2} = x_0 - 2$$

$$2 + \sqrt[3]{2} = x_0$$

$$(f^{-1})'(2) = \frac{1}{3(2+\sqrt[3]{2}-2)^2}$$

$$= \frac{1}{3(\sqrt[3]{2})^2}$$

$$= \boxed{\frac{1}{3\sqrt[3]{4}}}$$

Check by finding $f^{-1}(x)$ explicitly. [Sadly, this is not always possible.]

$$f(x) = (x-2)^3$$

$$x = (y-2)^3$$

$$\sqrt[3]{x} = y-2$$

$$2 + \sqrt[3]{x} = y$$

$$f^{-1}(x) = 2 + \sqrt[3]{x} \Rightarrow (f^{-1})'(x) = \frac{1}{3}x^{-2/3}$$

$$(f^{-1})'(2) = \frac{1}{3}(2)^{-2/3}$$

$$= \boxed{\frac{1}{3\sqrt[3]{4}}}$$

Math 250

Limit to a portion that passes HLT

(10) Find $(f^{-1})'(2)$ when $f(x) = \dfrac{x+3}{x+1}$, $x > -1$.

step 1: 2 is an x-coord on f^{-1}.

$$f^{-1}(2) = x_0$$
$$f(x_0) = 2$$
$$\frac{x_0+3}{x_0+1} = 2$$
$$x_0+3 = 2(x_0+1)$$
$$x_0+3 = 2x_0+2$$
$$1 = x_0$$

can be solved w/o GC!

(2,1) is on graph of f^{-1}
(1,2) is on graph of f.

step 2: $f'(x) = \dfrac{(x+1)\cdot 1 - (x+3)\cdot 1}{(x+1)^2}$ quotient rule

$$= \frac{x+1-x-3}{(x+1)^2}$$
$$= \frac{-2}{(x+1)^2}$$

step 3: Evaluate $f'(x)$ at an x-coordinate on graph of f.

$$f'(1) = \frac{-2}{(1+1)^2} = \frac{-2}{2^2} = \frac{-2}{4} = -\frac{1}{2}$$

step 4: $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{(-\frac{1}{2})} = \boxed{-2}$

The slope of the tangent line to $f^{-1}(x)$ at (2,1) is -2.
The slope of the tangent line to $f(x)$ at (1,2) is $-\frac{1}{2}$.

Math 250

11 Find $(f^{-1})'(2)$ when $f(x) = \sqrt{x-4}$

step 1: 2 is an x-coordinate on $f^{-1}(x) \Rightarrow (2, x_0)$ on f^{-1}

$\Rightarrow (x_0, 2)$ on f.

$f(x_0) = 2$

$\sqrt{x_0 - 4} = 2$ ← can be solved w/o GC

$x_0 - 4 = 4$

$x_0 = 8$

check for extraneous: $f(8) = \sqrt{8-4} = \sqrt{4} = 2 \checkmark$.

$(2, 8)$ on f^{-1}

$(8, 2)$ on f

step 2: $f'(x) = \frac{1}{2}(x-4)^{-1/2} \cdot 1 = \dfrac{1}{2\sqrt{x-4}}$

step 3: Evaluate $f'(x)$ at an x-coordinate on graph of f.

$f'(8) = \dfrac{1}{2\sqrt{8-4}} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2 \cdot 2} = \dfrac{1}{4}$

step 4: $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(8)} = \dfrac{1}{(1/4)} = \boxed{4}$

The slope of tangent line to f^{-1} at $(2, 8)$ is 4.

The slope of tangent line to f at $(8, 2)$ is $\frac{1}{4}$.

Math 250

Background Knowledge of Logs & Exponentials

- Relationship between the logarithm and exponential of the same base, composition
- Natural log, Common Log, Log of other base b
- Properties of logs: $\log_b(1)$ $\log_b(b^x)$
  $\log_b(x \cdot y)$
  $\log_b\left(\frac{x}{y}\right)$ $b^{\log_b x}$
  $\log_b x^k$
- Graph of logarithmic function, relationship to graph of exponential
- Domain, range of log and exponential functions and relationships
- e, the base of the natural logarithm and natural exponential
- asymptotes of logarithmic and exponential functions
- Solve equations containing exponentials
- Solve equations containing logs.
- Recognize the difference between an exponential and a polynomial.
- Change-of-base formula
- Recognize constants vs. variables e esp.

Math 250

Facts about Logs and Exponentials

1) $\underbrace{\log_b x = y}$ is equivalent to $\underbrace{b^y = x}$.

logarithmic form of equation $\leftrightarrow$ can be rewritten in either $\longleftrightarrow$ exponential form of equation

2) bases b must be $b > 0$ (continuous function)

$b \neq 1$ (function passes horizontal line test)

3) When the base b is the irrational number e,

$\log_e x$ is written $\ln x$, called natural logarithm.

4) When the base b is the number 10,

$\log_{10} x$ is written $\log x$, called common logarithm.

5) $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.

$f(g(x)) = x$ means $b^{\log_b x} = x$

$g(f(x)) = x$ means $\log_b b^x = x$

6) $b^0 = 1$ means $\log_b 1 = 0$

$10^0 = 1$ means $\log 1 = 0$

$e^0 = 1$ means $\ln 1 = 0$

Math 250

7) $b^{N+M} = b^N \cdot b^M$ means $\log_b x + \log_b y = \log_b(x \cdot y)$

exponent law — log property

8) $b^{N-M} = \dfrac{b^N}{b^M}$ means $\log_b x - \log_b y = \log_b\left(\dfrac{x}{y}\right)$

exponent law — log property

9) $b^{N \cdot k} = (b^N)^k$ means $k \cdot \log_b x = \log_b(x^k)$

exponent law — log property

10) change-of-base formula

$$\log_b x = \frac{\ln x}{\ln b} \quad \text{or} \quad \frac{\log x}{\log b} \quad \text{or} \quad \frac{\log_B x}{\log_B b}$$

for any valid base B, $B > 0$, $B \neq 1$.

11) There are no log properties for

$\log_b(x) \cdot \log_b(y)$ — product of two logs

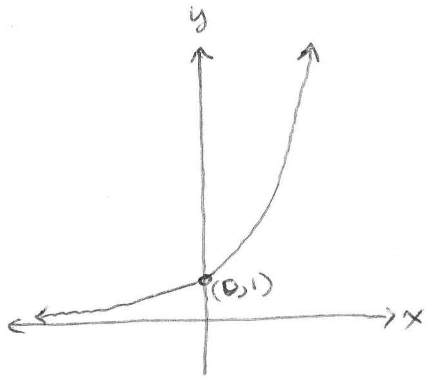
$\dfrac{\log_b(x)}{\log_b(y)}$ — quotient of two logs

$\log_b(x+y)$ — log of a sum

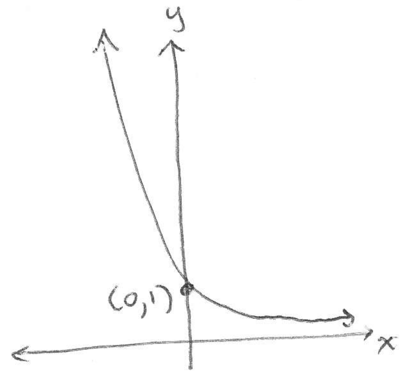
$\log_b(x-y)$ — log of a difference

Math 250 Background

12) Graph of $f(x) = b^x$



$b > 1$, including $b = e$



$0 < b < 1$

domain $x \in \mathbb{R}$

range $y > 0$

horizontal asymptote
$y = 0$

$b > 1$, including $b = e$:

$\lim_{x \to \infty} b^x = +\infty$

$\lim_{x \to -\infty} b^x = 0$

$\lim_{x \to 0} b^x = 1$

always increasing

$0 < b < 1$:

$\lim_{x \to \infty} b^x = 0$

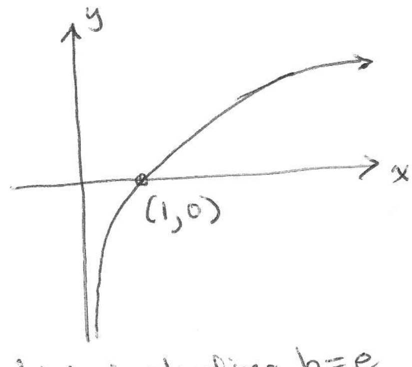
$\lim_{x \to -\infty} b^x = +\infty$

$\lim_{x \to 0} b^x = 1$

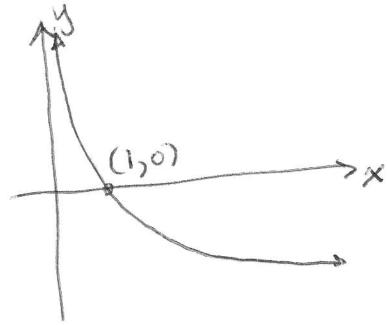
always decreasing

always concave up
always continuous
always differentiable

13) Graph of $f^{-1}(x) = \log_b x$



$b > 1$, including $b = e$



$0 < b < 1$

domain: $x > 0$

range: $y \in \mathbb{R}$

vertical asymptote
$x = 0$

$b > 1$, including $b = e$:

$\lim_{x \to \infty} \log_b x = +\infty$

$\lim_{x \to -\infty} \log_b x =$ DNE, function not defined

$\lim_{x \to 0^+} \log_b x = -\infty$

always increasing
always concave down

$0 < b < 1$:

$\lim_{x \to \infty} \log_b x = -\infty$

$\lim_{x \to -\infty} \log_b x =$ DNE, function not define

$\lim_{x \to 0^+} \log_b x = +\infty$

always decreasing
always concave up

continuous and differentiable $x > 0$

Math 250

SKIP

Proofs of Log Properties

5), 6) and 7) $\Rightarrow$ any base, Math 70/101.

Let $x = b^M$ and $y = b^N$. This means $\log_b x = M$ and $\log_b y = N$.

5) $\log_b(x \cdot y) = \log_b(b^M \cdot b^N)$ substitute.

$= \log_b(b^{M+N})$ property of exponents

$= M+N$ inverse functions

$= \log_b x + \log_b y$ substitute back.

$\therefore \log_b(xy) = \log_b x + \log_b y$

6) $\log_b\left(\frac{x}{y}\right) = \log_b\left(\frac{b^M}{b^N}\right)$ substitute

$= \log_b\left(b^{M-N}\right)$ property of exponents

$= M - N$ inverse functions

$= \log_b x - \log_b y$ subst back.

$\therefore \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

7) $\log_b(x^k) = \log_b((b^M)^k)$ substitute

$= \log_b(b^{M \cdot k})$ property of exponents

$= M \cdot k$ inverse functions

$= k \cdot M$ commutative property of multiplication

$= k \cdot \log_b x$ subst back.

$\therefore \log_b(x^k) = k \cdot \log_b(x)$.

Note: There are no expand / compress properties for

1) $\log_b(x) \cdot \log_b(y)$

2) $\log_b(x+y)$

3) $\dfrac{\log_b(x)}{\log_b(y)}$

4) $\log_b(x-y)$

Math 250 [redacted]

Expand each logarithm to the sum, difference and/or scalar multiple of logs.

① $\ln \frac{10}{9} = \boxed{\ln 10 - \ln 9}$

② $\ln \sqrt{3x+2} = \ln (3x+2)^{1/2} = \boxed{\frac{1}{2} \ln (3x+2)}$

③ $\ln \left(\frac{6x}{5}\right) = \ln (6x) - \ln 5 = \boxed{\ln 6 + \ln x - \ln 5}$

✓ ④ $\ln \frac{(x^2+3)^2}{x \sqrt[3]{x^2+1}} = \ln (x^2+3)^2 - \ln x \sqrt[3]{x^2+1}$

$= 2 \ln (x^2+3) - \left[\ln x + \ln \sqrt[3]{x^2+1}\right]$

$= \boxed{2 \ln (x^2+3) - \ln x - \frac{1}{3} \ln (x^2+1)}$

✓ ⑤ $\ln \sqrt{x^2(x+1)(x+2)} = \frac{1}{2} \ln \left[x^2 (x+1)(x+2)\right]$

$x>0$

$= \frac{1}{2} \left[\ln x^2 + \ln (x+1) + \ln (x+2)\right]$

$= \frac{1}{2} \ln x^2 + \frac{1}{2} \ln (x+1) + \frac{1}{2} \ln (x+2)$

$= \boxed{\ln x + \frac{1}{2} \ln (x+1) + \frac{1}{2} \ln (x+2)}$

✓ ⑥ $\ln \sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{2} \left[\ln (x^2-1) - \ln (x^2+1)\right]$

$x>1$

$= \frac{1}{2} \ln (x^2-1) - \frac{1}{2} \ln (x^2+1)$

$= \boxed{\frac{1}{2} \ln (x-1) + \frac{1}{2} \ln (x+1) - \frac{1}{2} \ln (x^2+1)}$

⑦ $\ln \frac{(x+1)(x-2)}{(x-1)(x+2)} = \boxed{\ln (x+1) + \ln (x-2) - \ln (x-1) - \ln (x+2)}$

$x>2$

⑧ $\ln \frac{x(x-1)^{3/2}}{\sqrt{x+1}} = \ln \left[x \cdot (x-1)^{3/2}\right] - \ln \sqrt{x+1}$

$= \ln x + \ln (x-1)^{3/2} - \ln (x+1)^{1/2}$

$x>1$

$= \boxed{\ln x + \frac{3}{2} \ln (x-1) - \frac{1}{2} \ln (x+1)}$

Math 250 Logarithm and Exponential Review problems from Math 70, Math 101, and Math 244

These are review questions you should be able to solve for exact answers without using your calculator.

Complete solutions are in the online notes.

Solve.

1) $16^{3/4} = 8^x$

2) $3\left(5^{x-1}\right) = 86$

3) $\log_2 3x = -\frac{1}{2}$

4) $\log_6(x+`1) + \log_6 x = 1$

5) $\log_7\left(x^2+7\right) = \frac{2}{3}\log_7 64$

6) $\log_b 81 = -4$

7) $\log_3 \frac{1}{81} = x$

8) $2x - 1 = \log_2 32$

9) $2x - 1 = \log_2 3$

Sketch graph, showing any asymptotes with dotted (dashed) lines.

10) $y = 3^{x-1}$

11) $y = \log_3(x-1)$

12) Use the change-of-base formula to write $\log_2 3$ using natural logarithms.

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Solutions to handout

$\boxed{\text{Solving Exponential Equations}}$ (Problem has an exponential)

Method 1: If the expressions can be written with the same base:

1 Solve $16^{3/4} = 8^x$

$\left.\begin{aligned} 16 &= 2^4 \\ 8 &= 2^3 \end{aligned}\right\}$ can both be written using base 2.

[Choose smaller bases for easier calculations.]

$(2^4)^{3/4} = (2^3)^x$ subst 2^4 for 16, 2^3 for 8

$2^{4\cdot\frac{3}{4}} = 2^{3x}$ mult exp

$2^3 = 2^{3x}$

$3 = 3x$ set exponents equal.

$\boxed{x=1}$ isolate variable

Method 2: If the expressions cannot be written with the same base, take logs (choose the base) of both sides.

2 $3(5^{x-1}) = 86$

$5^{x-1} = \frac{86}{3}$ Isolate the exponential first for a simpler outcome

$\ln(5^{x-1}) = \ln\left(\frac{86}{3}\right)$ Take logs both sides. I chose ln.

$(x-1)\cdot\ln(5) = \ln\left(\frac{86}{3}\right)$ Use log properties to bring exponent to front.

$x-1 = \dfrac{\ln\left(\frac{86}{3}\right)}{\ln(5)}$ Isolate x. Remember: ln(5) is just a number.

$\boxed{x = \dfrac{\ln\left(\frac{86}{3}\right)}{\ln(5)} + 1}$ ← exact answer

$\boxed{x \approx 3.0850}$ ← approximate answer

Solving Logarithmic Equations (Problem has a log.)

Method 1: When the argument is unknown.

3. $\log_2 3x = -\frac{1}{2}$ ← Given one log

$2^{(-1/2)} = 3x$ — Write in equivalent exponential form.

$\frac{1}{\sqrt{2}} = 3x$ — Isolate variable / Solve eqn + simplify

$x = \frac{1}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$\boxed{x = \frac{\sqrt{2}}{6}}$

4. $\log_6(x+1) + \log_6 x = 1$ ← Given several logs, same side

$\log_6(x \cdot (x+1)) = 1$ — Combine into a single log.

$\log_6(x^2 + x) = 1$

$6^1 = x^2 + x$ — write exponential form.

$x^2 + x - 6 = 0$ — Solve for x

$(x+3)(x-2) = 0$

$x = -3 \quad x = 2$ — * Check for extraneous *

$\log_6(-3+1) + \log_6(-3) = 1$

not real $\neq 1$

reject $x = -2$.

$\log_6(2+1) + \log_6(2) = 1$

$\log_6(3) + \log_6(2) = 1$

$\log_6(3 \cdot 2) = 1$

$6^1 = 6$ ✓

$\boxed{x = 2}$

5. $\log_7(x^2+7) = \frac{2}{3}\log_7 64$ ← Given logs on both sides and all terms have logs

$\log_7(x^2+7) = \log_7(64^{2/3})$ — Simplify to single log with coefficient 1 on each side.

$\log_7(x^2+7) = \log_7(16)$

Log equations, continued

(5) cont.

$$7^{\log_7(x^2+7)} = 7^{\log_7(16)}$$

Take corresponding exponential of both sides, i.e. set arguments =.

means $x^2+7=16$.

Solve resulting equation

$x^2-9=0$

$\boxed{x=\pm 3}$

Check for extraneous.

$\log_7(3^2+7)$ ok

$\log_7((-3)^2+7)$ ok.

Method 2: When base is unknown.

(6) $\log_b(81) = -4$

Write equivalent exponential.

Solve resulting equation

$b^{-4} = 81$

$\frac{1}{b^4} = 81$

$\frac{1}{81} = b^4$

$\pm\sqrt[4]{\frac{1}{81}} = b \Rightarrow b = \pm\frac{1}{3}$

Eliminate invalid bases $b>0$, $b\neq 1$

$\boxed{b=\frac{1}{3}}$

Method 3: When log value is unknown.

(7) $\log_3 \frac{1}{81} = x$

Write equivalent exponential, if necessary.

$3^x = \frac{1}{81}$

Solve as for exponential eqn. {Here, same base}.

$3^x = 3^{-4}$

$\boxed{x=-4}$

Method 4: When the log is a number in equation.

(8) $2x-1 = \log_2 32$

Resolve the log to a number, if possible.

$2x-1=5$

Solve eqn.

$2x=6$

$\boxed{x=3}$

Log equations, continued.

(9) $2x-1=\log_2 3$

If log does not resolve exactly, carry it as a simpler constant would be used.

$$2x=(\log_2 3)+1$$

Isolate x / Solve equation.

$$\boxed{x=\frac{\log_2(3)+1}{2}}$$ ← exact answer

$$x=\frac{\left(\frac{\ln 3}{\ln 2}\right)+1}{2}$$

Use change of base formula to calculate, if an approximate answer is needed.

$$x=\frac{\ln 3}{2\ln 2}+1$$

$$x=\frac{\ln 3}{\ln 4}+1$$ ← alternate form of exact answer. (slightly prettier).

$$\boxed{x\approx .7925}$$

Yes, you can write $1=\log_2 2$ and $\frac{1}{2}$ as exponent:

$$x=\frac{\log_2(3)+\log_2(2)}{2}$$

$$x=\frac{1}{2}\log_2(6)$$

$$\boxed{x=\log_2\sqrt{6}}$$

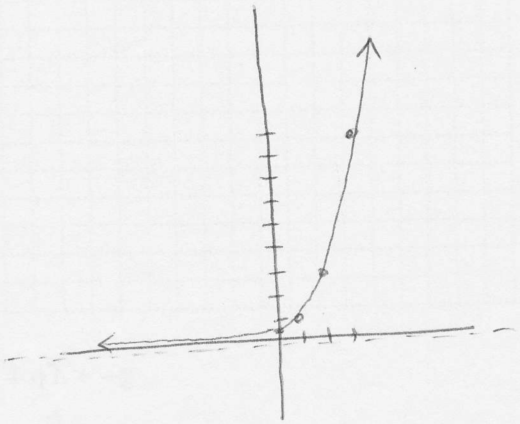
A very pretty answer! ☺

Graphs of Exponentials and Logs

(10) Sketch graph of $y = 3^{x-1}$.

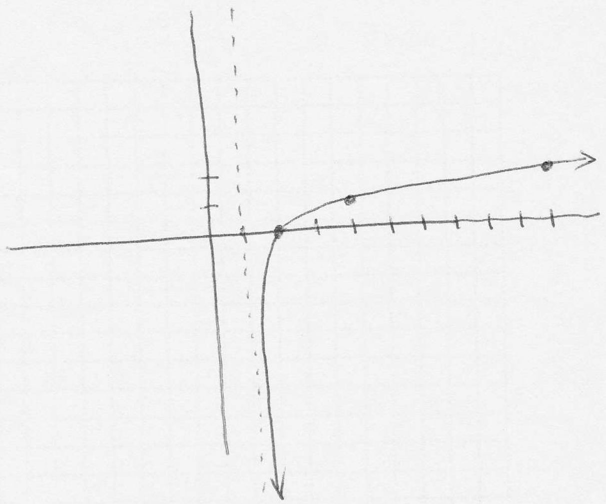
Show any asymptotes clearly.

If we know the graph of $f(x)$, the graph of $f(x-1)$ is shifted 1 unit right.



x	3^{x-1}
0	3^{-1}
1	3^{0}
2	3^{1}
3	3^{2}

(11) Sketch graph of $y = \log_3(x-1)$



POINTS ON $y = 3^x$

x	3^x
−2	3^{-2}
−1	3^{-1}
0	3^{0}
1	3^{1}
2	3^{2}
3	3^{3}

POINTS ON $y = \log_3 x$

x	$\log_3 x$
3^{-2}	−2
3^{-1}	−1
3^{0}	0
3^{1}	1
3^{2}	2
3^{3}	3

POINTS ON $y = \log_3(x-1)$

x	$\log_3(x-1)$
$3^{-1} + 1 = \frac{10}{9}$	−1
$2 = 3^{0} + 1$	0
$3^{1} + 1 = 4$	1
$3^{2} + 1 = 10$	2
$3^{3} + 1 = 28$	3

(12) Change-of-base formula

$$\log_2 3 = \boxed{\frac{\ln 3}{\ln 2}}$$

– END REVIEW –